On the concentration of interacting particle processes

Part I : Introduction and survey

P. Del Moral

INRIA & Bordeaux Mathematical Institute

Sino-French Summer Institute in Stochastic Modeling and Applications, Beijing, June 2011

Introduction

Stochastic particle methods
=
Universal adaptive sampling technique

2 types of stochastic interacting particle models:

- Diffusive particle models with mean field drifts
  [McKean-Vlasov style]
- Interacting jump particle models
  [Boltzmann & Feynman-Kac style]

Lectures $\subset$ Interacting jumps models

- **Interacting jumps = Recycling transitions =**
- **Discrete generation models ($\Leftrightarrow$ geometric jump times)**

Equivalent particle algorithms

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

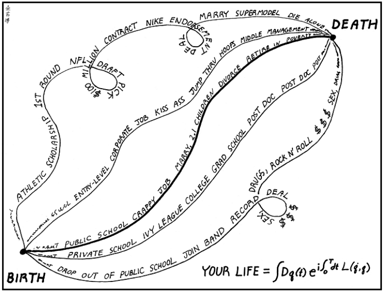
Equivalent particle algorithms

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching-selection
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

More botanical names: bootstrapping, spawning, cloning, pruning, replenish, enrichment, go with the winner, *some new ideas ?*

A single stochastic model

Particle interpretation of Feynman-Kac path integrals



The Path Integral Formulation of Your Life

Basic notation

$\mathcal{P}(E)$ probability meas., $\mathcal{B}(E)$ bounded functions on E.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $Q(x_1, dx_2)$ **integral operators** $x_1 \in E_1 \rightsquigarrow x_2 \in E_2$

$$\begin{aligned} Q(f)(x_1) &= \int Q(x_1, dx_2) f(x_2) \\ [\mu Q](dx_2) &= \int \mu(dx_1) Q(x_1, dx_2) \qquad (\Longrightarrow [\mu Q](f) = \mu[Q(f)]\ ) \end{aligned}$$

- **Boltzmann-Gibbs transformation**

  [Positive and bounded potential function G]

$$\mu(dx) \quad \mapsto \quad \Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

Basic notation

$E = \{1, \ldots, d\} \rightsquigarrow$ matrix operations

$$\mu = [\mu(1), \ldots, \mu(d)] \quad Q = (Q(i,j))_{1 \leq i,j \leq d} \quad f = \begin{bmatrix} f(1) \\ \vdots \\ f(d) \end{bmatrix}$$

and

$$\Psi_G(\mu) \propto \mu \ \mathrm{Diag}\,(G(1), \ldots, G(d))$$

Feynman-Kac measures

- Markov chain X_n on E_n, with transitions M_n :

$$\mathbb{P}_n\left(d(x_0,\ldots,x_n)\right) = \eta_0(dx_0)M_1(x_0,dx_1)\ldots M_n(x_{n-1},dx_n)$$

- Potential functions $G_n(x_n) \in [0,1]$

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n}\left\{\prod_{0\leq p<n} G_p(X_p)\right\}\, d\mathbb{P}_n$$

Flow of *n*-marginals

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f(X_n)\prod_{0\leq p<n} G_p(X_p)\right)$$

Nonlinear equation:

$$\eta_{n+1} = \Psi_{G_n}(\eta_n)M_{n+1} := \Phi_{n+1}(\eta_n)$$

2 "elementary" illustrations

- Confinements :

  Random walk $X_n \in \mathbb{Z}^d$, $X_0 = 0$ & $G_n := 1_{[-L,L]}$, $L > 0$.

$$\mathbb{Q}_n = \mathrm{Law}\left((X_0, \ldots, X_n) \mid X_p \in [-L, L],\ \forall 0 \leq p < n\right)$$

  and

$$\mathcal{Z}_n = \gamma_n(1) = \mathrm{Proba}\left(X_p \in [-L, L],\ \forall 0 \leq p < n\right)$$

- Self avoiding walks

$$\mathbf{X_n} = (X_0, \ldots, X_n) \quad \& \quad \mathbf{G_n}(\mathbf{X_n}) = 1_{X_n \notin \{X_0, \ldots, X_{n-1}\}}$$

$$\mathbb{Q}_n = \mathrm{Law}\left((\mathbf{X_0}, \ldots, \mathbf{X_n}) \mid X_p \neq X_q,\ \forall 0 \leq p < q < n\right)$$

  and

$$\mathcal{Z}_n = \gamma_n(1) = \mathrm{Proba}\left(X_p \neq X_q,\ \forall 0 \leq p < q < n\right)$$

$\supset$ *Continuous time models*

$$X_n := X'_{[t_n, t_{n+1}[} \quad \& \quad G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_s(X'_s) ds$$

$$\Downarrow$$

$$\prod_{0 \leq p < n} G_p(X_p) = \exp \left\{ \int_{t_0}^{t_n} V_s(X'_s) ds \right\}$$

or using a simple "Euler's scheme" $X'_{t_p} = X_p$

$$e^{\int_{t_0}^{t_n} \left[V_s(X'_s) ds + W_s(X'_s) dB_s \right]} \simeq \prod_{0 \leq p < n} e^{V_{t_p}(X_p)\ \Delta t + W_{t_p}(X_p) \sqrt{\Delta t}\ N_p(0,1)}$$

The 3 keys formulae

- Time marginal measures = Path space measures:

$$\gamma_n(f_n) = \mathbb{E}\left(f_n(\mathbf{X_n}) \prod_{0\leq p<n} \mathbf{G_p}(\mathbf{X_p})\right)$$

$$[\ \mathbf{X_n} := (X_0, \ldots, X_n)\ \ \&\ \ \mathbf{G_n}(\mathbf{X_n}) = G_n(X_n)]\ \Longrightarrow\ \eta_n = \mathbb{Q}_n$$

- Normalizing constants (= Free energy models):

$$\mathcal{Z}_n = \mathbb{E}\left(\prod_{0\leq p<n} G_p(X_p)\right) = \prod_{0\leq p<n} \eta_p(G_p)$$

The last key

- Backward Markov models

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_n(x_{n-1},dx_n)$$

with

$$\begin{array}{rcl} Q_n(x_{n-1},dx_n) & := & G_{n-1}(x_{n-1})M_n(x_{n-1},dx_n) \\ & \stackrel{hyp}{=} & H_n(x_{n-1},x_n)\ \nu_n(dx_n) \\ & \Rightarrow & \eta_{n+1}(dx) = \dfrac{1}{\eta_n(G_n)}\ \eta_n\left(H_{n+1}(\centerdot,x)\right)\ \nu_{n+1}(dx) \end{array}$$

If we set

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) = \frac{\eta_n(dx_n)\ H_{n+1}(x_n,x_{n+1})}{\eta_n\left(H_{n+1}(\centerdot,x_{n+1})\right)}$$

then we find the backward equation

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) = \frac{1}{\eta_n(G_n)}\ \eta_n(dx_n)\ Q_{n+1}(x_n,dx_{n+1})$$

The last key (continued)

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \eta_0(dx_0)Q_1(x_0,dx_1)\ldots Q_n(x_{n-1},dx_n)$$

$\oplus$

$$\eta_{n+1}(dx_{n+1})\ \mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ Q_{n+1}(x_n,dx_{n+1})$$

$$\Downarrow$$

Backward Markov chain model :

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n)\ \mathbb{M}_{n,\eta_{n-1}}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0}(x_1,dx_0)$$

with the dual/backward <u>Markov</u> transitions

$$\mathbb{M}_{n+1,\eta_n}(x_{n+1},dx_n) \propto \eta_n(dx_n)\ H_{n+1}(x_n,x_{n+1})$$

A bad tempting idea = i.i.d. weighted samples X_n^i

$$\mathcal{Z}_n := \mathbb{E}\left(\prod_{0\leq p<n} G_p(X_p)\right) \simeq \mathcal{Z}_n^N := \frac{1}{N}\sum_{i=1}^{N}\prod_{0\leq p<n} G_p(X_p^i)$$

Example :
X_n simple RW $\in \mathbb{Z}^d$, $G_n = 1_{[-10,10]} \Rightarrow \exists n = n(\omega)\ :\ \mathcal{Z}_n^N = 0$

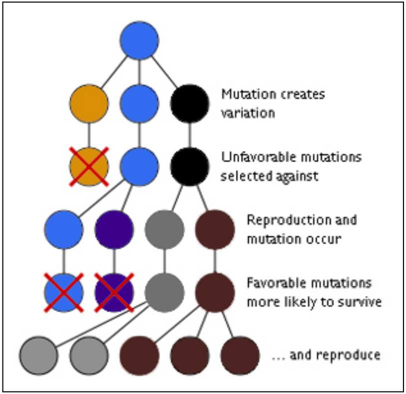
$$N\ \mathbb{E}\left(\left[\frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} - 1\right]^2\right) = \frac{1-\mathcal{Z}_n}{\mathcal{Z}_n} \simeq \mathrm{Proba}(X_p \in A,\ \forall 0 \leq p < n)^{-1}$$

Our objective

$$N\ \mathbb{E}\left(\left[\frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} - 1\right]^2\right) \leq c \times n$$

Solution = Genetic style interacting jumps

- Mutation-Exploration-Proposals w.r.t. transitions M_n.
- Selection-Recycling-(Accept.-Rejet) w.r.t. potential G_n.



Stochastic population dynamics

$$
\left.\begin{array}{c} \xi_n^1 \\ \vdots \\ \xi_n^i \\ \vdots \\ \xi_n^N \end{array}\right]
\xrightarrow{\;S_{n,\eta_n^N}\;}
\left[\begin{array}{ccc} \widehat{\xi}_n^1 & \xrightarrow{\;M_{n+1}\;} & \xi_{n+1}^1 \\ \vdots & & \vdots \\ \widehat{\xi}_n^i & \longrightarrow & \xi_{n+1}^i \\ \vdots & & \vdots \\ \widehat{\xi}_n^N & \longrightarrow & \xi_{n+1}^N \end{array}\right]
$$

Accept/Reject-Recycle=Selection =Geometric recycling interacting jumps

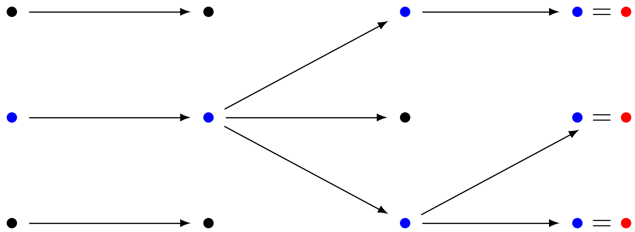
$$S_{n,\eta_n^N}(\xi_n^i, dx)$$

$$:= G_n(\xi_n^i)\ \delta_{\xi_n^i}(dx) + \left(1 - G_n(\xi_n^i)\right)\ \sum_{j=1}^N \frac{G_n(\xi_n^j)}{\sum_{k=1}^N G_n(\xi_n^k)} \delta_{\xi_n^j}(dx)$$

Ex. : $G_n = 1_A \rightsquigarrow G_n(\xi_n^i) = 1_A(\xi_n^i)$

The 4 particle estimates

Genealogical tree evolution $(N, n) = (3, 3)$

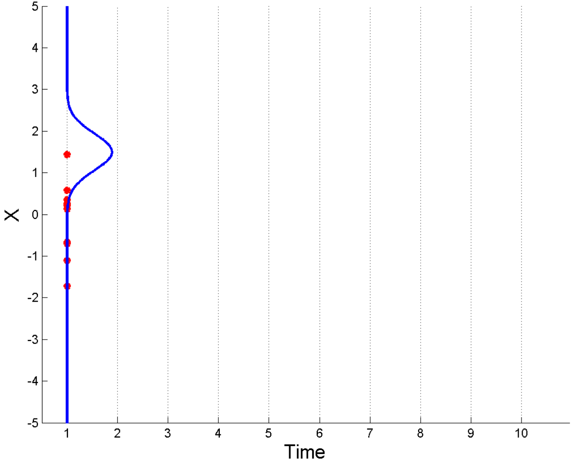


- Individuals in the current population

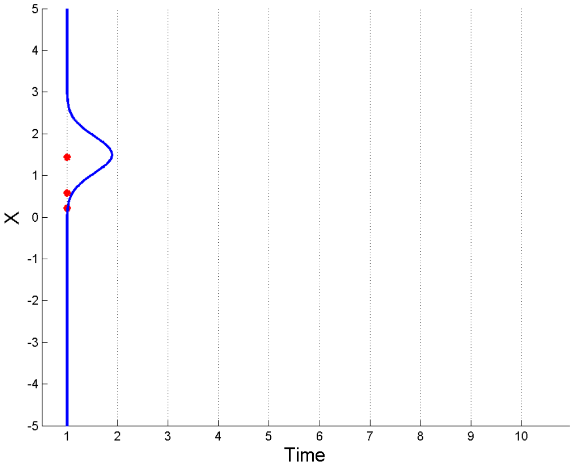
  $=$ *Almost* i.i.d. samples w.r.t. FK marginal meas. η_n

$$\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \longrightarrow_{N \to \infty} \eta_n = \textbf{solution of a nonlinear m.v.p.}$$

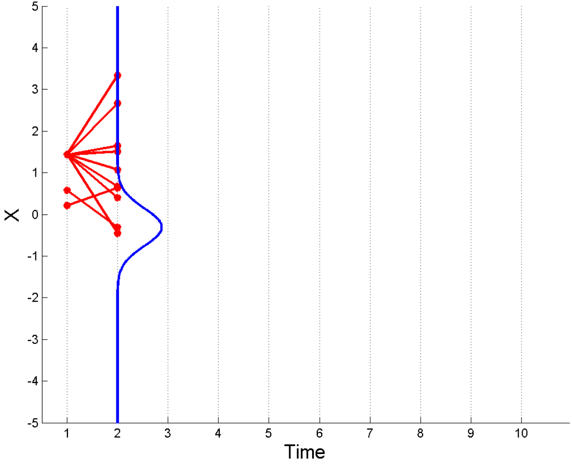
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



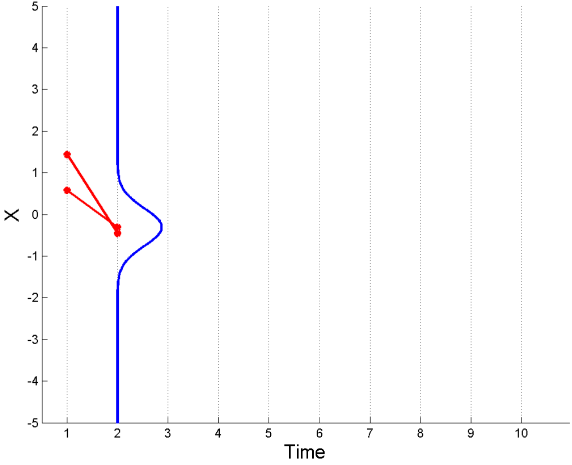
Graphical illustration : $\eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \simeq \eta_n$



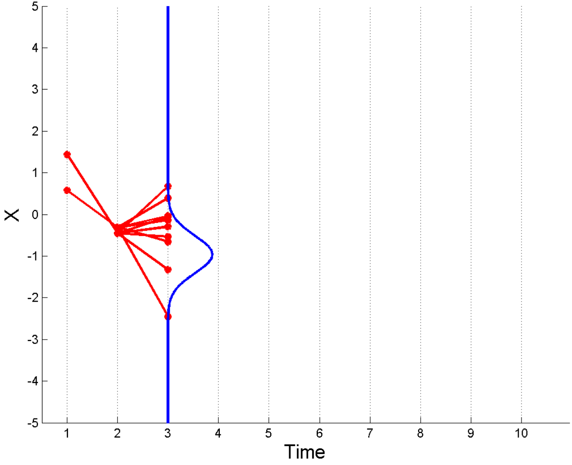
Graphical illustration : $\eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \simeq \eta_n$



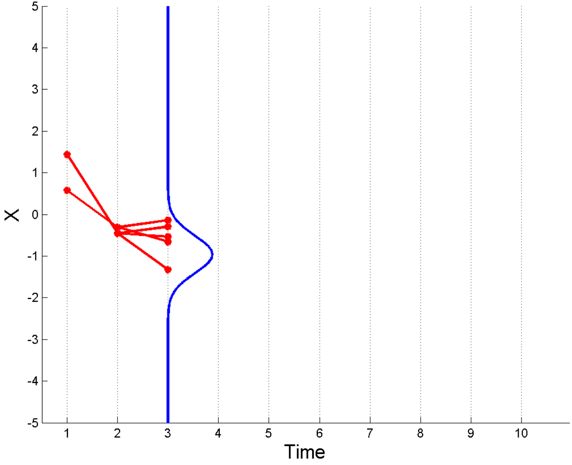
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



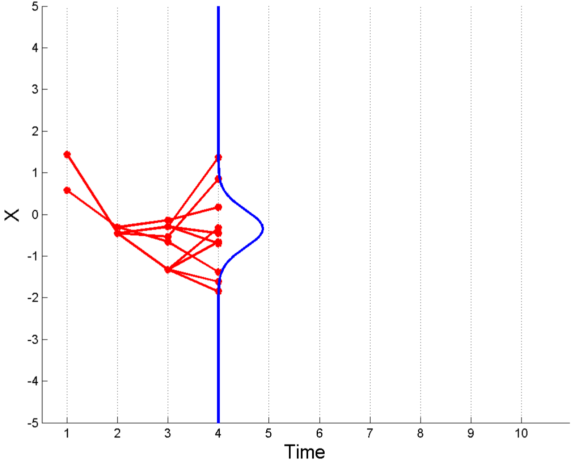
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\xi_n^i} \simeq \eta_n$



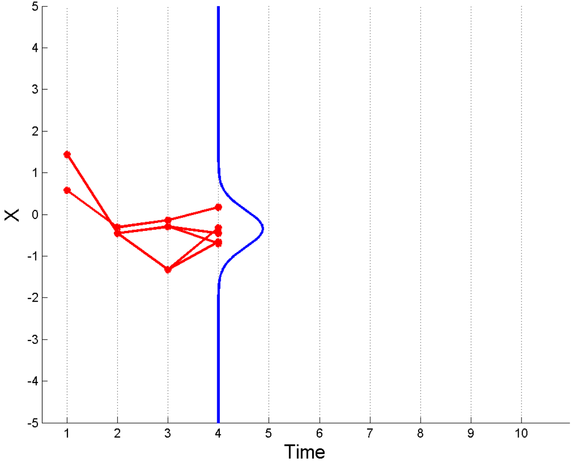
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



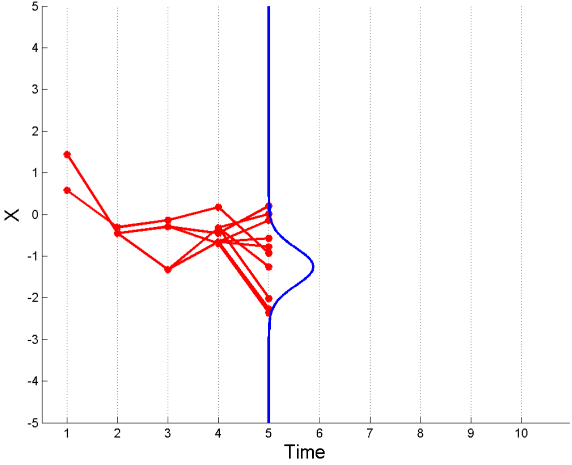
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



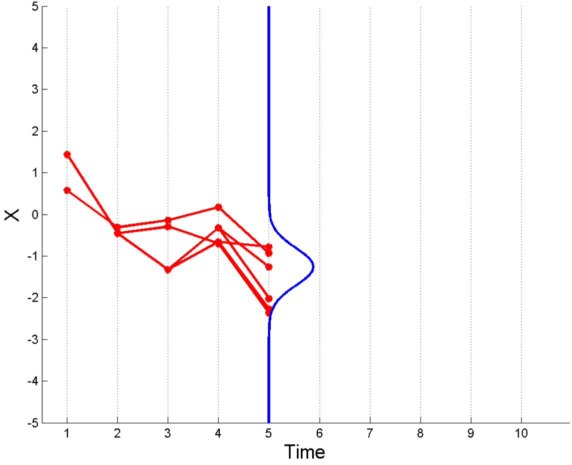
Graphical illustration : $\eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \simeq \eta_n$



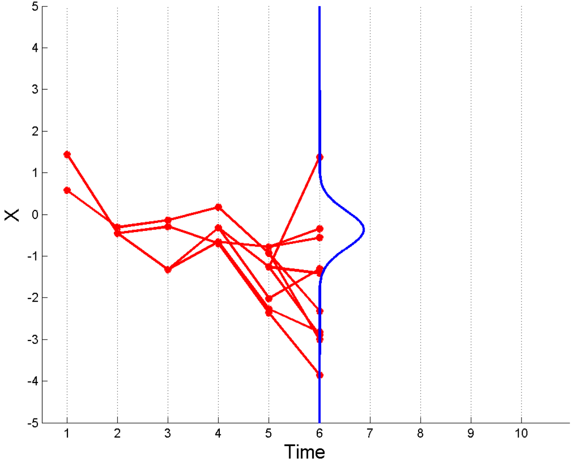
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



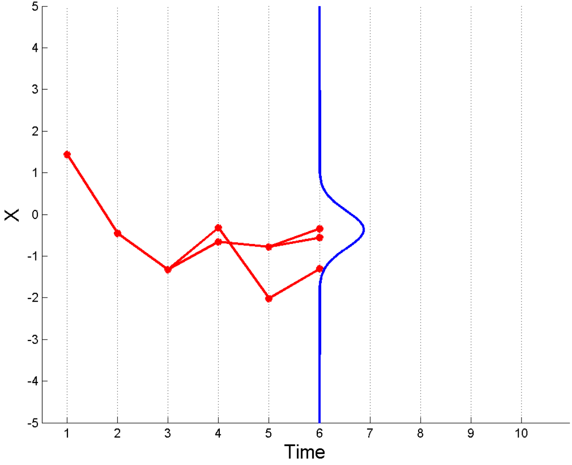
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



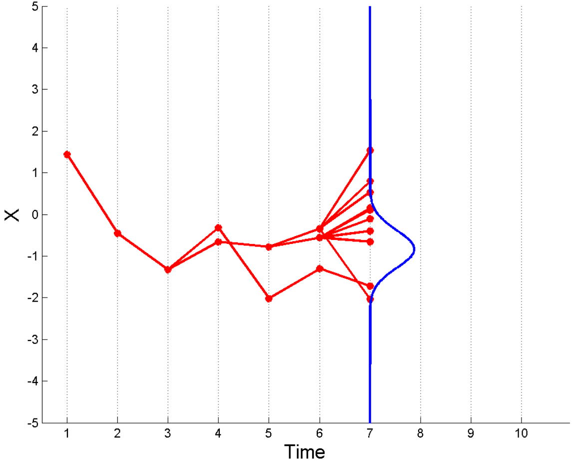
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



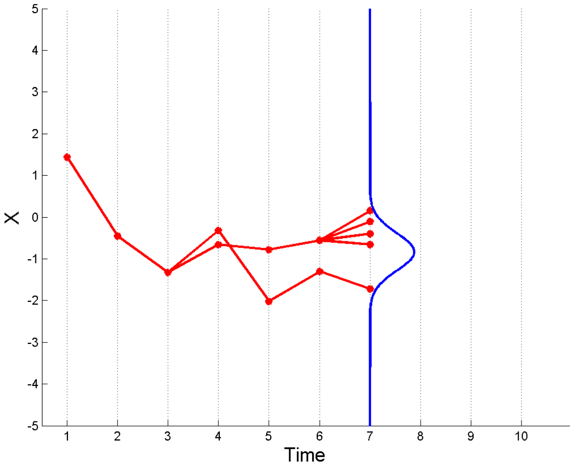
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



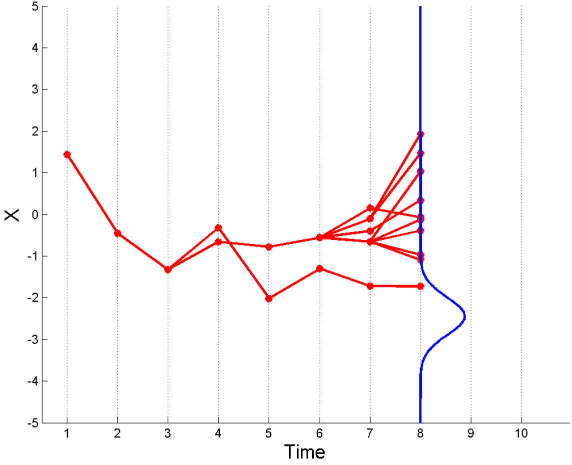
Graphical illustration : $\eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \simeq \eta_n$



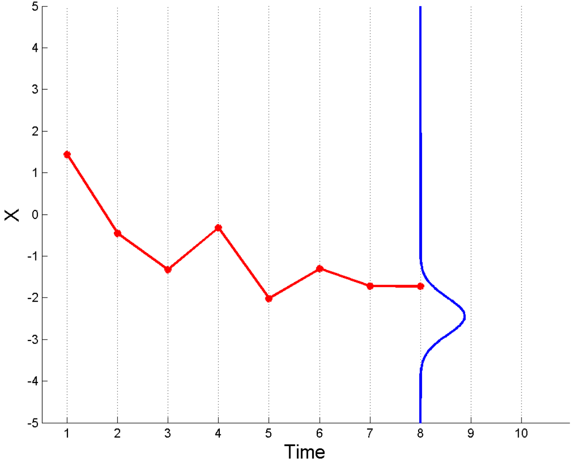
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\xi_n^i} \simeq \eta_n$



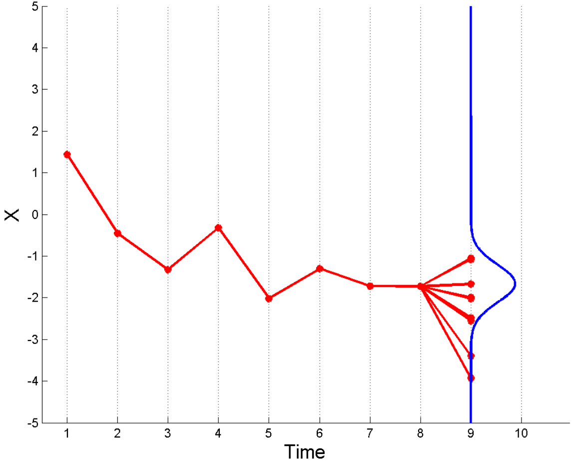
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



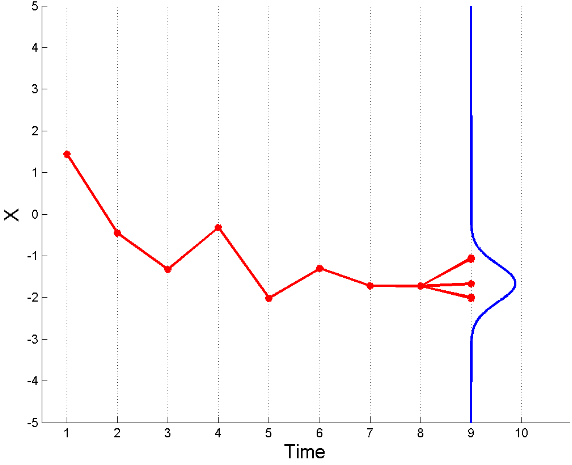
Graphical illustration : $\eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i} \simeq \eta_n$



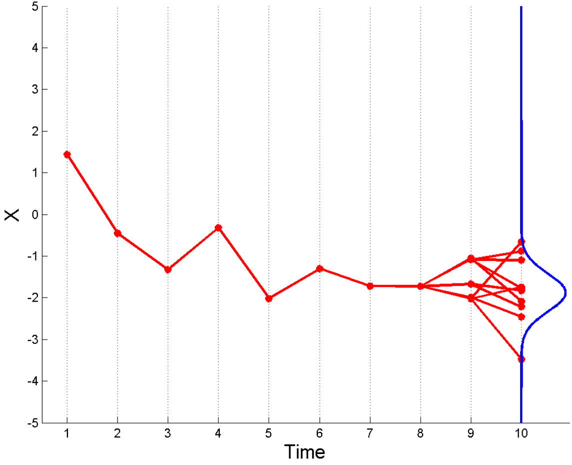
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



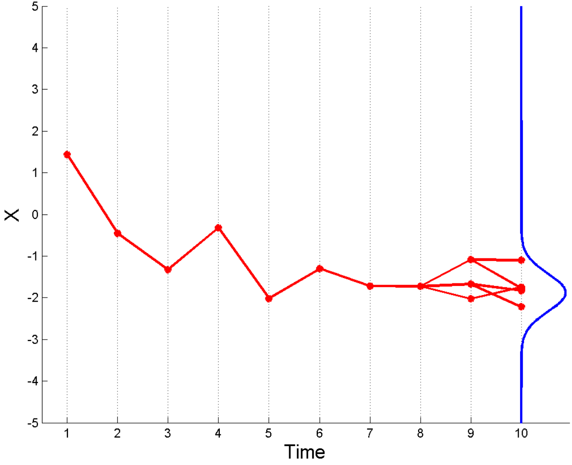
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



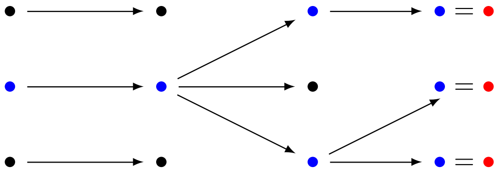
Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



Graphical illustration : $\eta_n^N := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i} \simeq \eta_n$



Two more particle estimates



- Ancestral lines = *Almost* i.i.d. sampled paths w.r.t. $\mathbb{Q}_n$.

$$\left(\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}\right) := i\text{-th ancestral line} \quad i\text{-th current individual} = \xi^i_n$$

$$\frac{1}{N} \sum_{1 \le i \le N} \delta_{\left(\xi^i_{0,n}, \xi^i_{1,n}, \ldots, \xi^i_{n,n}\right)} \longrightarrow_{N \to \infty} \mathbb{Q}_n$$

- Unbiased particle free energy functions

$$\mathcal{Z}^N_n = \prod_{0 \le p < n} \eta^N_p(G_p) \longrightarrow_{N \to \infty} \mathcal{Z}_n = \prod_{0 \le p < n} \eta_p(G_p)$$

... and the last particle measure

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) := \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0^N}(x_1,dx_0)$$

with the random particle matrices:

$$\mathbb{M}_{n+1,\eta_n^N}(x_{n+1},dx_n) \propto \eta_n^N(dx_n)\ H_{n+1}(x_n,x_{n+1})$$

Example: Normalized additive functionals

$$\mathbf{f_n}(x_0,\ldots,x_n) = \frac{1}{n+1}\sum_{0\leq p\leq n} f_p(x_p)$$

$$\Downarrow$$

$$\mathbb{Q}_n^N(\mathbf{f_n}) := \frac{1}{n+1}\sum_{0\leq p\leq n} \eta_n^N\,\underbrace{\mathbb{M}_{n,\eta_{n-1}^N}\ldots\mathbb{M}_{p+1,\eta_p^N}(f_p)}_{\text{matrix operations}}$$

Island models

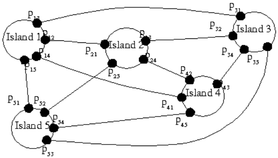


Fig 3.4 Schematic of a genetic algorithm using island migration

Reminder : the unbiased property

$$\mathbb{E}\left(\mathbf{f_n}(\mathbf{X_n}) \prod_{0\leq p<n} \mathbf{G_p}(\mathbf{X_p})\right) = \mathbb{E}\left(\eta_n^N(\mathbf{f_n}) \prod_{0\leq p<n} \eta_p^N(\mathbf{G_p})\right)$$

$$= \mathbb{E}\left(\mathbf{F_n}(\mathcal{X}_n) \prod_{0\leq p<n} \mathcal{G}_p(\mathcal{X}_p)\right)$$

with the Island evolution Markov chain model

$$\mathcal{X}_n := \eta_n^N \qquad \text{and} \quad \mathcal{G}_n(\mathcal{X}_n) = \eta_n^N(\mathbf{G_n}) = \mathcal{X}_n\left(\mathbf{G_n}\right)$$

$\Rightarrow$ $\boxed{\text{particle model with } (\mathcal{X}_n, \mathcal{G}_n(\mathcal{X}_n)) = \text{Interacting Island particle model}}$

Current population models

Constants (c_1, c_2) related to (bias,variance), c universal constant
Test funct. $\|f_n\| \leq 1$, $\forall \ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

  $$\left[\eta_n^N - \eta_n\right](f) \leq \frac{c_1}{N} \ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}} \ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $x = (x_i)_{1 \leq i \leq d} \rightsquigarrow (-\infty, x] = \prod_{i=1}^d (-\infty, x_i]$ cells in $E_n = \mathbb{R}^d$.

  $$F_n(x) = \eta_n\left(1_{(-\infty,x]}\right) \quad \text{and} \quad F_n^N(x) = \eta_n^N\left(1_{(-\infty,x]}\right)$$

  The probability of the following event

  $$\sqrt{N} \ \left\|F_n^N - F_n\right\| \leq c \ \sqrt{d \ (x+1)}$$

  is greater than $1 - e^{-x}$.

Particle free energy models

Constants (c_1, c_2) related to (bias,variance), c universal constant
$\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- Unbiased property

$$\mathbb{E}\left(\eta_n^N(f_n) \prod_{0\leq p<n} \eta_p^N(G_p)\right) = \mathbb{E}\left(f_n(X_n) \prod_{0\leq p<n} G_p(X_p)\right)$$

- For any $\epsilon \in \{+1, -1\}$, the probability of the event

$$\frac{\epsilon}{n} \log \frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

is greater than $1 - e^{-x}$.

note $(0 \leq \epsilon \leq 1 \Rightarrow (1 - e^{-\epsilon}) \vee (e^{\epsilon} - 1) \leq 2\epsilon)$

$$e^{-\epsilon} \leq \frac{z^N}{z} \leq e^{\epsilon} \Rightarrow \left|\frac{z^N}{z} - 1\right| \leq 2\epsilon$$

Genealogical tree models $:= \eta_n^N$ (in path space)

Constants (c_1, c_2) related to (bias,variance), c universal constant
$\mathbf{f_n}$ test function $\|\mathbf{f_n}\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \mathbb{Q}_n\right](f) \leq c_1\ \frac{n+1}{N}\ \left(1 + x + \sqrt{x}\right) + c_2\ \sqrt{\frac{(n+1)}{N}}\ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $\mathcal{F}_n =$ indicator fct. $\mathbf{f_n}$ of cells in $\mathbf{E_n} = \left(\mathbb{R}^{d_0} \times \ldots, \times \mathbb{R}^{d_n}\right)$
  The probability of the following event

$$\sup_{\mathbf{f_n} \in \mathcal{F}_n} \left|\eta_n^N(\mathbf{f_n}) - \mathbb{Q}_n(\mathbf{f_n})\right| \leq c\ (n+1)\ \sqrt{\frac{\sum_{0 \leq p \leq n} d_p}{N}\ (x+1)}$$

  is greater than $1 - e^{-x}$.

Backward particle models

Constants (c_1, c_2) related to (bias,variance), c universal constant.
$\mathbf{f_n}$ normalized additive functional with $\|f_p\| \leq 1$, $\forall \; (x \geq 0, n \geq 0, N \geq 1)$

- The probability of the event

$$\left[\mathbb{Q}_n^N - \mathbb{Q}_n\right](\overline{\mathbf{f}}_n) \leq c_1 \; \frac{1}{N} \; (1 + (x + \sqrt{x})) + c_2 \; \sqrt{\frac{x}{N(n+1)}}$$

- $\mathbf{f}_{a,n}$ normalized additive functional w.r.t. $f_p = 1_{(-\infty, a]}$, $a \in \mathbb{R}^d = E_n$
  .
  The probability of the following event

$$\sup_{a \in \mathbb{R}^d} \left|\mathbb{Q}_n^N(\mathbf{f_{a,n}}) - \mathbb{Q}_n(\mathbf{f_{a,n}})\right| \leq c \; \sqrt{\frac{d}{N}(x+1)}$$

  is greater than $1 - e^{-x}$.

Some ideas on how it works

Feynman-Kac evolution equations

$$\eta_n \xrightarrow{\text{Correction-Selection}} \widehat{\eta}_n = \Psi_{G_n}(\eta_n) \xrightarrow{\text{Prediction-Mutation}} \eta_{n+1} = \widehat{\eta}_n M_{n+1}$$

with the nonlinear mass transport equation

$$\Psi_{G_n}(\eta_n)(dy) := \frac{1}{\mu(G_n)} \; G_n(y) \; \eta_n(dy) = \int \eta_n(dx) S_{n,\eta_n}(x, dy)$$

with

$$S_{n,\eta_n}(x, .) := G_n(x) \; \delta_x + (1 - G_n(x)) \;\; \Psi_{G_n}(\eta_n)$$

$$\Downarrow$$

$$\eta_{n+1} = \eta_n \left(S_{n,\eta_n} M_{n+1} \right) := \eta_n K_{n+1,\eta_n} = \mathrm{Law}(\overline{X}_n)$$

McKean Markov chain model $\overline{X}_n$ = Perfect sampler

η_n = Law of $\overline{X}_n$ = **nonlinear Markov chain**

$$\mathbb{P}(\overline{X}_{n+1} \in dx \mid \overline{X}_n) = K_{n+1,\eta_n}(\overline{X}_n, dx)$$

Mean field particle model = Stochastic population dynamic model

$\rightsquigarrow$ $\xi_n = (\xi_n^1, \ldots, \xi_n^N) \in E_n^N$ **such that**

$$\eta_n^N := \frac{1}{N} \sum_{1 \le i \le N} \delta_{\xi_n^i} \simeq_{N \uparrow \infty} \eta_n$$

$\rightsquigarrow$ Natural approximated transitions

$$\xi_n^i \rightsquigarrow \xi_{n+1}^i \sim K_{n+1,\eta_n^N}(\xi_n^i, dx)$$

Some key advantages

- Mean field models=Stochastic linearization/perturbation technique

$$\eta_n^N = \eta_{n-1}^N K_{n,\eta_{n-1}^N} + \frac{1}{\sqrt{N}}\ W_n^N$$

  with $W_n^N \simeq W_n$ independent centered Gaussian fields .

- $\eta_n = \eta_{n-1} K_{n,\eta_{n-1}}$ stable $\Rightarrow$ Non propagation of local sampling errors

  $\Longrightarrow$ Uniform control w.r.t. the time horizon

- "No burning, no need to study the stability of MCMC models".
- Stochastic adaptive grid approximation
- Nonlinear system $\rightsquigarrow$ positive - beneficial interactions.
- Simple and natural sampling algorithm.
- Local conditional iid samples $\oplus$ Stability of nonlinear sg

  $\rightsquigarrow$ New concentration and empirical process theory